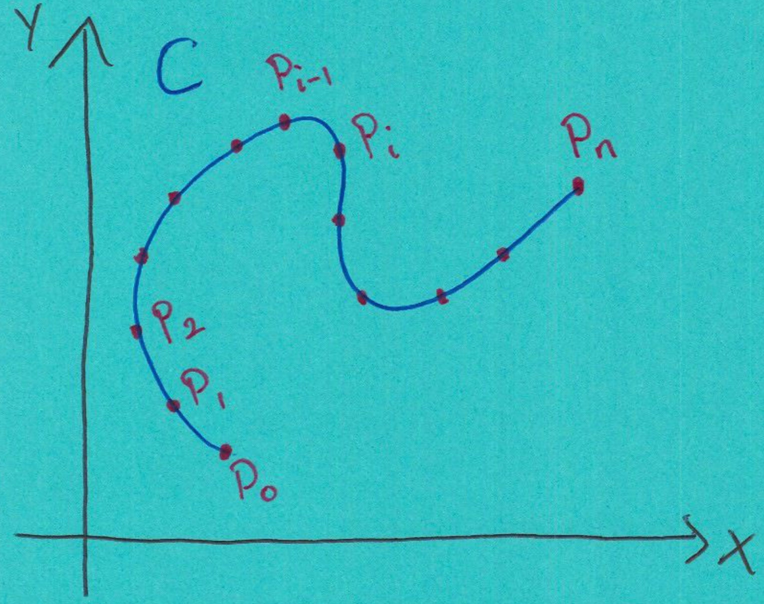
Lecture 31

16.2 - Line Integrals (Part I)

Suppose we have a curve C, parametrized by $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$, in $\mathbb{R}^2$ which is smooth, i.e., $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t) \neq \vec{0}$. Suppose $f(x,y)$ is a function whose domain includes C, then; can we compute the integral of f along C,

$$\int_C f(x,y)\,ds \quad ?$$

We define the line integral as follows:



Break the interval $[a,b]$ into n pieces: $[t_{i-1}, t_i]$ and let $P_i = \vec{r}(t_i)$. This breaks C into n pieces of (arc) length Δs_i. Inside each interval $[t_{i-1}, t_i]$ we choose a sample point t_i^*, which gives us sample points along the curve $(x_i^*, y_i^*) = \vec{r}(t_i^*)$

We define

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if the limit exists. What is ds? In words, it's the infinitesimal arclength of C. Recall the arclength function $s(t) = \int_a^t |\vec{r}'(u)|\,du$

$$\Rightarrow \frac{ds}{dt} = |\vec{r}'(t)| \Rightarrow ds = |\vec{r}'(t)|\,dt$$

So, we can compute the line integral as

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}'(t)|\,dt = \int_a^b f(x(t),y(t))\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt}$$

Ex: Compute the integral of $f(x,y) = 18y^3$ along the piece of the curve $x = y^3$ from $(-1,-1)$ to $(1,1)$.

Sol: First, we parametrize our curve: if we let $y = t$, then $x = t^3$, so $\vec{r}(t) = \langle t^3, t\rangle$. As for the bounds: $\vec{r}(t) = \langle -1,-1\rangle$ when $t=-1$ & $\vec{r}(t) = \langle 1,1\rangle$ when $t=1$, so $-1 \le t \le 1$. Now $|\vec{r}'(t)| = |\langle 3t^2, 1\rangle| = \sqrt{9t^4+1}$, so

$$\int_C 18y^3\,ds = \int_{-1}^{1} 18(t)^3\sqrt{9t^4+1}\,dt \overset{u=9t^4+1}{=} \frac{1}{2}\int_{10}^{10}\sqrt{u}\,du = 0. \quad \diamond$$

The integral $\int_C f(x,y)\,ds$ is sometimes called a scalar line integral or an integral with respect to arclength.

If C is a piecewise-smooth curve, i.e., a collection, $C_1, \cdots, C_n$, of smooth curves joined end on end, then we compute

$$\int_C f\,ds = \int_{C_1} f\,ds + \cdots + \int_{C_n} f\,ds$$

What is a physical interpretation of $\int_C f\,ds$?

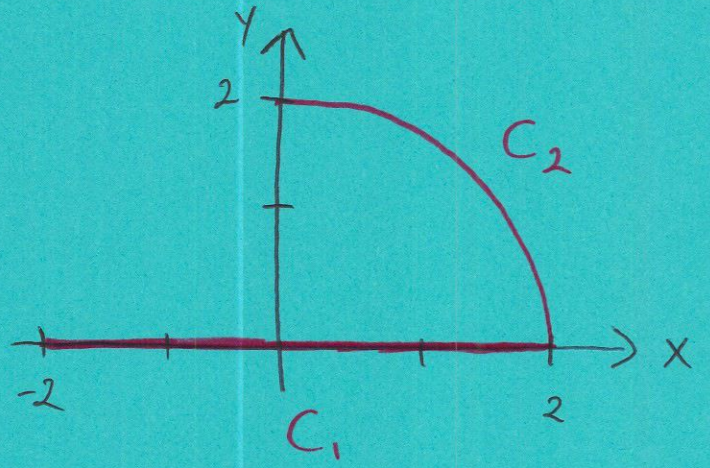
Suppose we choose f to be the linear density function $\rho(x,y)$ of a thin wire bent in the shape of C. Then, the mass, m, of the wire is: $\boxed{m = \int_C \rho(x,y)\,ds}$

and the center of mass of the wire has coordinates $(\bar{x}, \bar{y})$ given by:

$$\bar{x} = \frac{1}{m}\int_C x\rho(x,y)\,ds \quad , \quad \bar{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds$$

Ex A wire is bent in the shape of a hook which looks like the line segment from $(-2,0)$ to $(2,0)$ followed by the piece of the circle of radius 2 in the first quadrant. If the linear density of the wire is given by $\rho(x,y)=\frac{1}{2}$. Find the coordinates of the center of mass of the hook.

Sol: The hook looks like:



Call the pieces C_1 and C_2. First, we parametrize C.

C_1: $\vec{r}_1(t)=\langle t,0\rangle$, $-2\le t\le 2$

C_2: $\vec{r}_2(t)=\langle 2\cos t, 2\sin t\rangle$, $0\le t\le \frac{\pi}{2}$

Let's first find the mass:

$$m=\int_C \rho\, ds=\int_{C_1}\rho\, ds+\int_{C_2}\rho\, ds=\int_{-2}^{2}\frac{1}{2}|\vec{r}_1'(t)|\,dt+\int_0^{\frac{\pi}{2}}\frac{1}{2}|\vec{r}_2'(t)|\,dt$$

$$=\int_{-2}^{2}\frac{1}{2}dt+\int_0^{\frac{\pi}{2}}dt=2+\frac{\pi}{2}=\frac{4+\pi}{2}$$

The integrals for the coordinates of the center of mass are:

$$\bar{x} = \frac{1}{m}\int_C x\rho\,ds = \frac{1}{m}\left[\int_{C_1} x\rho\,ds + \int_{C_2} x\rho\,ds\right]$$

$$= \frac{1}{m}\left[\int_{-2}^{2}(t)\left(\tfrac{1}{2}\right)(|\vec{r}_1{}'(t)|)\,dt + \int_0^{\frac{\pi}{2}}(2\cos t)\left(\tfrac{1}{2}\right)(|\vec{r}_2{}'(t)|)\,dt\right]$$

$$= \frac{1}{m}\left[\int_{-2}^{2} \frac{t}{2}\,dt + \int_0^{\frac{\pi}{2}} 2\cos t\,dt\right] = \frac{1}{m}\left[0+2\right] = \frac{4}{4+\pi}$$

$$\bar{y} = \frac{1}{m}\int_C y\rho\,ds = \frac{1}{m}\left[\int_{C_1} y\rho\,ds + \int_{C_2} y\rho\,ds\right]$$

$$= \frac{1}{m}\left[\int_{-2}^{2}(0)\left(\tfrac{1}{2}\right)(|\vec{r}_1{}'(t)|)\,dt + \int_0^{\frac{\pi}{2}}(2\sin t)\left(\tfrac{1}{2}\right)(|\vec{r}_2{}'(t)|)\,dt\right]$$

$$= \frac{1}{m}\left[0 + \int_0^{\pi/2} 2\sin t\,dt\right] = \frac{1}{m}\left[0+(0-(-2))\right] = \frac{4}{4+\pi}$$

So, $(\bar{x},\bar{y}) = \left(\frac{4}{4+\pi}, \frac{4}{4+\pi}\right)$ ◇

We can also define line integrals along curves C in $\mathbb{R}^3$. If C is parametrized by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$ and C is smooth, then:

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt = \int_a^b f(x(t),y(t),z(t))\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt}$$